Matryoshkan *d*-space[1] explorations…

Question

How to find the maximum number of segments any given space is divided in after 'cutting it up' any number of times..?

Underlying proposition

'Space' is matryoshkan in nature: space of dimension *d* contains space of dimension (*d* – 1), and is contained in space of dimension (*d* + 1).

Cutting up 0-space

Imagine a zero-dimensional space (or 0-space), a mathematical point. As you may know, a mathematical point is infinitesimally small, and thus indivisible. So, trivially, the maximum number of segments in which a 0-space can be divided, is 1…

$$\mathbf{S_0(\mathit{c}) = 1}$$

Cutting up 1-space

Now imagine a one-dimensional space (or 1-space), a mathematical line. Dividing this space is possible, by cutting it by a point. In other words, a cut in 1-space is a 0-space.
Each cut c divides the space into the number of segments it already had, plus 1, or in general terms: $S(c) = S(c-1) + 1$
Since $S(c-1) = S(c-2) + 1$, $S(c)$ can be rewritten as $S(c) = S(c-2) + 1 + 1$, and consequently $S(c) = S(c-3) + 1 + 1 + 1$, and so on, expanding until the smallest sensible term is reached at the first cut $S(1) = S(0) + 1$, in which $S(0)$, the uncut original line would of course equal 1….
$S(c) = (1 + 1 + 1 + \ldots + 1 + 1 + 1) + 1$
$\Rightarrow S(c) = c \times 1 + 1$
$\Rightarrow S(c) = c + 1$

$$\mathbf{S_1(\mathit{c}) = \mathit{c} + 1}$$

[1] Here, '*d*-space' is to be understood as an open, unbounded *d*-dimensional space.

Cutting up 2-space

Let's move on to a 2-space, a plane. Dividing this space is possible by cutting it by a line. In other words, a cut in 2-space is a 1-space.[2]
Here, the (maximum) number of segments for each cut c equals the number of cuts plus the (maximum) number of segments before the last cut, or in general terms: $S(c) = c + S(c-1)$.
Since $S(c-1) = (c-1) + S(c-2)$, $S(c)$ can be rewritten as $S(c) = c + (c-1) + S(c-2)$, and consequently $S(c) = c + (c-1) + (c-2) + S(c-3)$, and so on, expanding until the smallest sensible term is reached at the first cut $S(1) = 1 + S(0)$, in which $S(0)$, the uncut original surface would of course equal 1….
So, $S(c) = 1 + 1 + 2 + 3 + \ldots + (c-2) + (c-1) + c$.
Since this is 1 + the sum of an arithmetic sequence (in which c is the number of terms, 1 is the smallest term and c is the largest term), it can be rewritten as

$$S(c) = 1 + \frac{c(1+c)}{2}$$

$$\Rightarrow S(c) = \frac{c^2 + c + 2}{2}$$

$$\mathbf{S_2(c) = \frac{c^2 + c + 2}{2}}$$

Cutting up *d*-space

Now, before we tackle 3-space and beyond, let's take a look at a different approach in solving S_0, S_1 and S_2.

If we sum up the number of segments that S_0, S_1 and S_2 are divided in, for $c > 0$, we get

$c =$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$S_0(c) = 1$	1	1	1	1	1	1	1	1	1	1	1
$S_1(c) = c + 1$	1	2	3	4	5	6	7	8	9	10	11
$S_2(c) = \frac{c^2 + c + 2}{2}$	1	2	4	7	11	16	22	29	37	46	56

There's something interesting to be seen here…

[2] A 'cut' in d-space is made by a $(d-1)$-space.

$c =$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$S_0(c) = 1$	1	1	1	1	1	1	1	1	1	1	1
$S_1(c) = c + 1$	1	2	3	4	5	6	7	8	9	10	11
$S_2(c) = \frac{c^2 + c + 2}{2}$	1	2	4	7	11	16	22	29	37	46	56

As you can see, $S_1(3) = S_1(2) + S_0(2)$, and $S_2(6) = S_2(5) + S_1(5)$, and $S_2(9) = S_2(8) + S_1(8)$.
Or, in general: $S_d(c) = S_d(c-1) + S_{d-1}(c-1)$

Since $S_d(0) = 1$ for all d, we can derive $S_d(c)$ for any d and c, simply by filling in the adjacent numbers..!

Table 1. Maximum number of segments in *d*-space, after *c* cuts

$c =$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$S_0(c)$	1	1	1	1	1	1	1	1	1	1	1
$S_1(c)$	1	2	3	4	5	6	7	8	9	10	11
$S_2(c)$	1	2	4	7	11	16	22	29	37	46	56
$S_3(c)$	1	2	4	8	15	26	42	64	93	130	176
$S_4(c)$	1	2	4	8	16	31	57	99	163	256	386

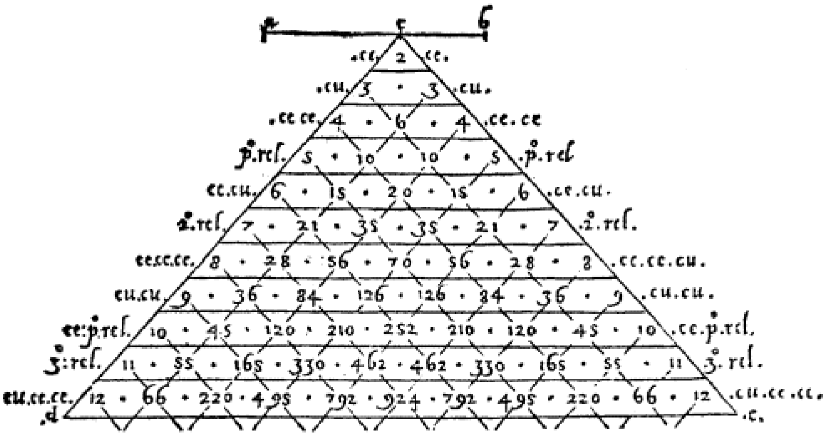
etc.

Kewl..!

But, what if you want to know the maximum number of segments in forty-space when cutting it 23 times..? Finding out by using this method would be a bit cumbersome, wouldn't it..? Creating a huge table, and performing a lot of additions… So, how about some formulas..?

Okay… But first, let's pay a visit to Blaise Pascal…

In his book *Traité du triangle arithmétique*, Pascal wrote about some of the properties of the arithmetical triangle, which was already known and studied for centuries by Indian, Persian and Chinese mathematicians.



From this triangle, all binomial coefficients can be derived, for which the principle goes

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$, for every positive integer k and n, where $k \le n$.

Also, $\binom{n}{k} = \binom{n}{n-k}$

Another way of representing the triangle, showing both principles, would be

Table 2. Pascal's triangle

n=	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
k = 0	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10
2			1	3	6	10	15	21	28	36	45
3				1	4	10	20	35	56	84	120
4					1	5	15	35	70	126	210
5						1	6	21	56	126	252
6							1	7	28	84	210
7								1	8	36	120
8									1	9	45
9										1	10
10											1

It looks like the numbers in light and dark blue in Table 1 and Table 2 are similarly connected. Maybe it's possible to find a common factor, and find a way to represent the first by means of the second…

Let's take a closer look at both tables…

n =	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
k = 0	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10
2			1	3	6	10	15	21	28	36	45
3				1	4	10	20	35	56	84	120
4					1	5	15	35	70	126	210
5						1	6	21	56	126	252
6							1	7	28	84	210
7								1	8	36	120
8									1	9	45
9										1	10
10											1

c =	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$S_0(c)$	1	1	1	1	1	1	1	1	1	1	1
$S_1(c)$	1	2	3	4	5	6	7	8	9	10	11
$S_2(c)$	1	2	4	7	11	16	22	29	37	46	56
$S_3(c)$	1	2	4	8	15	26	42	64	93	130	176
$S_4(c)$	1	2	4	8	16	31	57	99	163	256	386

What can be seen here, is

$$S_3(7) = \binom{7}{3} + \binom{7}{2} + \binom{7}{1} + \binom{7}{0}, \text{ and } S_4(10) = \binom{10}{4} + \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0}$$

Or, in general, $S_d(c) = \binom{c}{d} + \binom{c}{d-1} + \binom{c}{d-2} + \ldots + \binom{c}{2} + \binom{c}{1} + \binom{c}{0}$

So, $S_d(c) = \sum_{i=0}^{d} \binom{c}{i}$, or $S_d(c) = S_{d-1}(c) + \binom{c}{d}$

In the case of, for example, $S_4(2)$, $S_3(2)$ and $S_2(2)$, it can also be observed that no value is added with subsequent higher dimensions, which makes sense, since for every $c < d$, $\binom{c}{d}$ has no value.

So when $d > c$, the above formula can be replaced with

$$S_d(c) = \binom{c}{c} + \binom{c}{c-1} + \binom{c}{c-2} + \ldots + \binom{c}{2} + \binom{c}{1} + \binom{c}{0}$$

Now, this sequence has a very special property…

The *binomial theorem* states, that when a binomial like $x + y$ is raised to a positive integer power we get:
$(x + y)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n$,
where the binomial coefficients a_i (hence the name…) are the numbers in column n of Pascal's triangle. For example, for $n = 4$ we get

$$(x + y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$\Rightarrow (x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Now, if we substitute $x = 1$ and $y = 1$ we get the following:

$$(1 + 1)^4 = \binom{4}{0}1^4 + \binom{4}{1}1^3 \cdot 1 + \binom{4}{2}1^2 \cdot 1^2 + \binom{4}{3}1 \cdot 1^3 + \binom{4}{4}1^4$$

$$\Rightarrow 2^4 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$$

This, of course, goes for all $n > 0$.

So, in our case, $\binom{c}{c} + \binom{c}{c-1} + \binom{c}{c-2} + \ldots + \binom{c}{2} + \binom{c}{1} + \binom{c}{0} = 2^c$

Based on this, the general binomial formula can be simplified further.

Let’s take a look at, for example, $S_6(9)$…

binomial coefficients $n =$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$k = 0$	1	1	1	1	1	1	1	1	1	*1*	1
1		1	2	3	4	5	6	7	8	*9*	10
2			1	3	6	10	15	21	28	*36*	45
3				1	4	10	20	35	56	84	120
4					1	5	15	35	70	126	210
5						1	6	21	56	126	252
6							1	7	28	84	210
7								1	8	36	120
8									1	9	45
9										1	10
10											1

As we now know, $S_9(9) = 2^9$, so $S_6(9) = 2^9 - \binom{9}{7} - \binom{9}{8} - \binom{9}{9}$

Since $\binom{n}{k} = \binom{n}{n-k}$,we know that $\binom{9}{7} + \binom{9}{8} + \binom{9}{9} = \binom{9}{2} + \binom{9}{1} + \binom{9}{0}$

And since $\binom{9}{2} + \binom{9}{1} + \binom{9}{0} = S_2(9)$, we can rewrite the above equation as

$S_6(9) = 2^9 - S_2(9)$

This leads us to the general rule $S_d(c) = 2^c - S_{(d')}(c)$, where $d' = c - d - 1$

Also, if c is an odd number, another special case can be observed.

See for example $S_3(7)$

Since $S_3(7) = \binom{7}{3} + \binom{7}{2} + \binom{7}{1} + \binom{7}{0}$,

and $\binom{7}{7} + \binom{7}{6} + \binom{7}{5} + \binom{7}{4} + \binom{7}{3} + \binom{7}{2} + \binom{7}{1} + \binom{7}{0} = 2^c$

and given $\binom{7}{7} + \binom{7}{6} + \binom{7}{5} + \binom{7}{4} = \binom{7}{3} + \binom{7}{2} + \binom{7}{1} + \binom{7}{0}$

it follows that $S_3(7) = \frac{2^7}{2} = 2^7 \cdot 2^{-1} = 2^6$

Or, in general: if $d = ½(c - 1)$, then $S_d(c) = 2^{c-1}$

So, there are four ways to calculate $S_d(c)$, depending on the relative value of c and d, as is depicted in Table 3.

Table 3. Calculating $S_d(c)$, depending on relative value of c and d

$c > d$	$½(c - 1) > d$	$\mathbf{S_{d-1}(c) + \binom{c}{d}}$	
	$½(c - 1) = d$	$\mathbf{2^{c-1}}$	
	$½(c - 1) < d$	$\mathbf{2^c - S_{(d')}(c)}$	where $d' = c - d - 1$
$c \leq d$		$\mathbf{2^c}$	

In this way, the number of calculations one has to perform never exceeds half the number of cuts.

But still, there may be an even quicker way of getting things done. What we need to have is a single formula for each and every dimension…

Okay, how to go about this matter..? Let's return to Pascal…

In what is known as *Pascal's rule*, it is stated that

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$, where $a! = a \times (a - 1) \times (a - 2) \times \ldots 3 \times 2 \times 1$

For example,

$$\binom{9}{4} = \frac{9!}{(9-4)!4!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{(5\times4\times3\times2\times1)\times(4\times3\times2\times1)} = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 3\times7\times6 = 126$$

And for binomials with one variable

$$\binom{c}{0} = \frac{c!}{(c-0)!0!} = \frac{c\times(c-1)\times(c-2)\times(c-3)\times\ldots\times3\times2\times1}{(c\times(c-1)\times(c-2)\times(c-3)\times\ldots\times3\times2\times1)\times(1)} = 1$$

where 0! = 1 by default.

$$\binom{c}{1} = \frac{c!}{(c-1)!1!} = \frac{c\times(c-1)\times(c-2)\times(c-3)\times\ldots\times3\times2\times1}{((c-1)\times(c-2)\times(c-3)\times\ldots\times3\times2\times1)\times(1)} = \frac{c}{1} = c$$

$$\binom{c}{2} = \frac{c!}{(c-2)!2!} = \frac{c\times(c-1)\times(c-2)\times(c-3)\times\ldots\times3\times2\times1}{((c-2)\times(c-3)\times\ldots\times3\times2\times1)\times(2\times1)} = \frac{c\times(c-1)}{2} = \frac{c^2-c}{2}$$

$$\binom{c}{3} = \frac{c!}{(c-3)!3!} = \frac{c\times(c-1)\times(c-2)\times(c-3)\times\ldots\times3\times2\times1}{((c-3)\times(c-4)\times\ldots\times3\times2\times1)\times(3\times2\times1)}$$

$$= \frac{c\times(c-1)\times(c-2)}{6} = \frac{c^3-3c^2+2c}{6}$$

As you may have seen already, a shorter way of writing $\binom{c}{3}$ would be

$\binom{c}{3} = \binom{c}{2}\times\frac{c-2}{3}$, or in general $\binom{c}{k} = \binom{c}{k-1}\times\frac{c-(k-1)}{k}$

$$\binom{c}{4} = \binom{c}{3}\times\frac{c-3}{4} = \frac{(c^3-3c^2+2c)\times(c-3)}{6\times4} = \frac{c^4-6c^3+11c^2-6c}{24}$$

$$\binom{c}{5} = \binom{c}{4}\times\frac{c-4}{5} = \frac{(c^4-6c^3+11c^2-6c)\times(c-4)}{24\times5} = \frac{c^5-10c^4+35c^3-50c^2+24c}{120}$$

etc.

Now, remembering that $S_d(c) = \sum_{i=0}^{d} \binom{c}{i}$, we can derive the formula for the maximum number of segments for any *d*-space.

Table 4. Polynomials for the maximum number of sections after *c* cuts in zero- through seven-spaces

$S_0(c)$ =	$\binom{c}{0}$	1
$S_1(c)$ =	$S_0(c) + \binom{c}{1}$	$c + 1$
$S_2(c)$ =	$S_1(c) + \binom{c}{2}$	$\frac{c^2 + c + 2}{2}$
$S_3(c)$ =	$S_2(c) + \binom{c}{3}$	$\frac{c^3 + 5c + 6}{6}$
$S_4(c)$ =	$S_3(c) + \binom{c}{4}$	$\frac{c^4 - 2c^3 + 11c^2 + 14c + 24}{24}$
$S_5(c)$ =	$S_4(c) + \binom{c}{5}$	$\frac{c^5 - 5c^4 + 25c^3 + 5c^2 + 94c + 120}{120}$
$S_6(c)$ =	$S_5(c) + \binom{c}{6}$	$\frac{c^6 - 9c^5 + 55c^4 - 75c^3 + 304c^2 + 444c + 720}{720}$
$S_7(c)$ =	$S_6(c) + \binom{c}{7}$	$\frac{c^7 - 14c^6 + 112c^5 - 350c^4 + 1099c^3 + 364c^2 + 3828c + 5040}{5040}$
…	…	…
$S_d(c)$ =	$\sum_{i=0}^{d} \binom{c}{i}$	…

As is to be expected, calculations get lengthier with increasing values of *d*.
But maybe a shortcut can be found…

Being polynomials, each of the above equations is of the form
$a_d c^d + a_{d-1}c^{d-1} + a_{d-2}c^{d-2} + \ldots + a_2c^2 + a_1c^1 + a_0c^0$

It's tantalising to think that, by looking at the values of a_i, some pattern or other related to *d* and *i* can be revealed…

Table 5. The coefficient a_i

$i =$	0	1	2	3	4	5	6	7
$d = 0$	1							
1	1	1						
2	2	1	1					
3	6	5	0	1				
4	24	14	11	–2	1			
5	120	94	5	25	–5	1		
6	720	444	304	–75	55	–9	1	
7	5040	3828	364	1099	–350	112	–14	1

A few patterns do emerge (one of which I never would have found if I hadn't been roaming the caverns of N.J.A. Sloan's *On-Line Encyclopedia of Integer Sequences* at http://www.research.att.com/~njas/sequences/..!)

Quite obvious are $a_d = 1$, and $a_0 = d!$

And there's also $a_0 = \sum_{i=1}^{d} a_i$ (or: $a_1 + a_2 + \ldots + a_{i-1} + a_i$, for $1 \le i \le d$)

$i =$	0	1	2	3	4	5	6	7
$d = 0$	1							
1	1	1						
2	2	1	1					
3	6	5	0	1				
4	24	14	11	–2	1			
5	120	94	5	25	–5	1		
6	720	444	304	–75	55	–9	1	
7	5040	3828	364	1099	–350	112	–14	1

But how about $a_1 = d!\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{(-1)^d}{d-1}+\frac{(-1)^{d+1}}{d}\right)$, Sloan A024167

i =	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
d = *0*	1							
1	1	1						
2	2	1	1					
3	6	5	0	1				
4	24	14	11	–2	1			
5	120	94	5	25	–5	1		
6	720	444	304	–75	55	–9	1	
7	5040	3828	364	1099	–350	112	–14	1

Or $a_{d-1} = \frac{-d^2+3d}{2}$

i =	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
d = *0*	1							
1	1	1						
2	2	1	1					
3	6	5	0	1				
4	24	14	11	–2	1			
5	120	94	5	25	–5	1		
6	720	444	304	–75	55	–9	1	
7	5040	3828	364	1099	–350	112	–14	1

But that's about all I could find; more patterns are looming – e.g. a_{d-2}, above in yellow – but I haven't cracked 'em yet… Maybe a general 'shortcut' algorithm to derive *all* formulas, if indeed such an algorithm exists, is hiding itself from me, but perhaps some of the smarter (or more persistent) minds in our midst will be able to find it. The first thing to do, though, would be expanding the number of formulas, up to $S_{12}(c)$ for example, so a more robust set of numbers is available for crunching. Anyway, an interesting challenge, me reckons..!

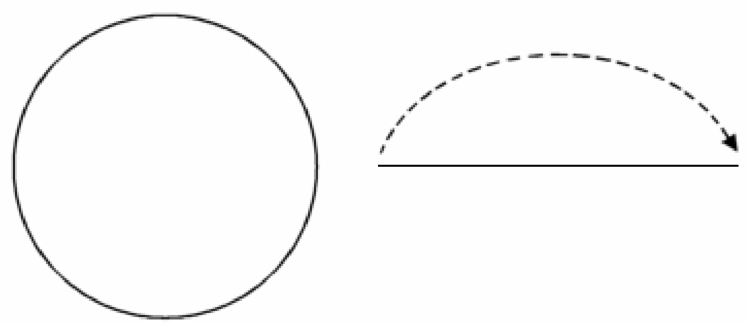
But even without the 'shortcut' algorithm, one needn't always waste too much time in finding the maximum number of segments for any number of cuts, in any given space, given the various calculation methods shown in Table 3… Remember cutting up forty-space 26 times..? Well, you won't get more than 2^{26} segments… A mere 67,108,864…

Cutting up tori in *d*-space

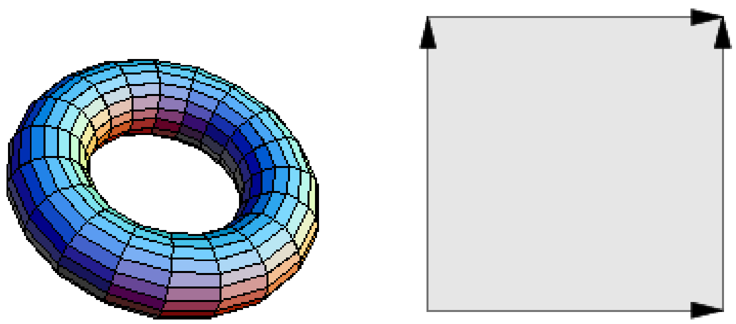
A torus is a compact manifold, meaning it is closed and bounded, and every point has a neighbourhood which resembles Euclidean space.

In a one-dimensional manifold, every point has a neighbourhood that looks like a segment of a line, for examples a line and a circle. In a two-dimensional manifold, every point has a neighbourhood that looks like a disk, for example a plane, the surface of a sphere, and the surface of a torus.

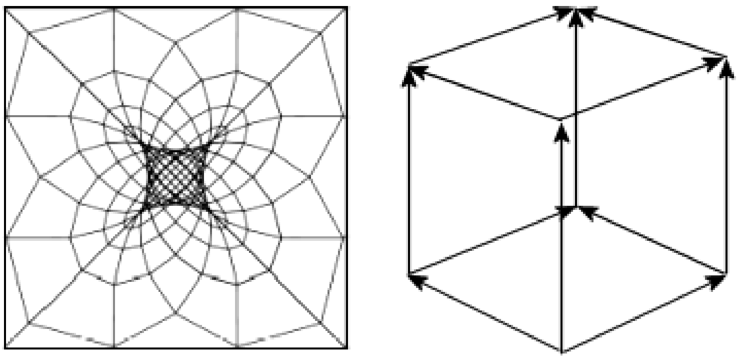
A one-dimensional torus (or 1-torus) can be constructed by connecting the opposite ends of a closed and bounded segment of 1-space (a line segment), where the connection is a point (i.e. 0-space).



A two-dimensional torus (or 2-torus) can be constructed by connecting the opposite sides of a closed and bounded segment of 2-space (e.g. a square), where each connection is a line (i.e. 1-space)



A 3-torus can be constructed by connecting the opposite faces of a closed and bounded segment of 3-space (e.g. a cube), where each connection is a plane (i.e. 2-space)

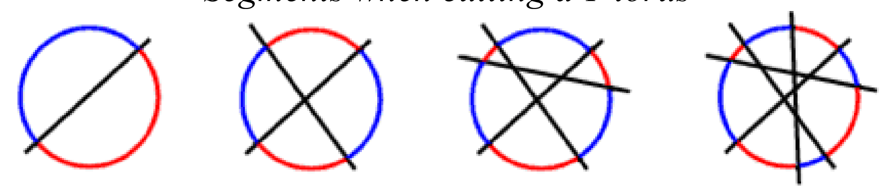


In general, a *d*-torus can be obtained from a *d*-dimensional hypercube by connecting the opposite faces (where each is a closed and bounded $(d-1)$-space), resulting in a closed and bounded manifold of *d*-space, extended into $(d+1)$-space.

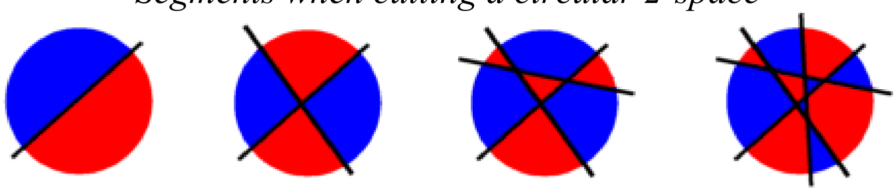
Contrary to cutting a *d*-space, which is done by a $(d-1)$-space, cutting a *d*-torus is done by a *d*-space. Although a *d*-torus is a *d*-space manifold, its extensiveness is in $(d+1)$-space. And cutting in a $(d+1)$-space is done by a *d*-space…

So, although a 1-torus is a circular 1-space, and not a circular 2-space, it is still cut by lines.

Segments when cutting a 1-torus



Segments when cutting a circular 2-space



The maximum number of segments obtained from cutting a 1-torus seems pretty straight forward: it's double the number of cuts. $T_1(c) = 2c$, for $c > 0$.

When combining this sequence with the sequence of the maximum number of segments from cutting a 2-torus, given by the formula $T_2(c) = \frac{c^3 + 5c}{3}$, it may be possible to find a common factor, which is likely to be linked to the binomial coefficients…

$c =$	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$T_1(c)$	2	4	6	8	10	12	14	16	18	20
$T_2(c)$	2	6	14	28	50	82	126	184	258	350

binomial coefficients $n =$	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$k = 1$	1	2	3	4	5	6	7	8	9	10
2		1	3	6	10	15	21	28	36	45
3			1	4	10	20	35	56	84	120
4				1	5	15	35	70	126	210
5					1	6	21	56	126	252
6						1	7	28	84	210
7							1	8	36	120
8								1	9	45
9									1	10
10										1

What can be seen here, is $T_1(4) = 2\cdot\binom{4}{1}$, and $T_2(5) = 2\cdot\binom{5}{1} + 2\cdot\binom{5}{2} + 2\cdot\binom{5}{3}$.

Or, in general $T_1(c) = 2\cdot\binom{c}{1}$, and $T_2(c) = 2\cdot\binom{c}{1} + 2\cdot\binom{c}{2} + 2\cdot\binom{c}{3}$, for $c > 0$

Although a mere two sets of numbers is a very small (or even, no) basis to go from, one might be inclined to see a pattern here.

If one rewrites $T_2(c)$ as $T_1(c)+2\cdot\binom{c+1}{3}$, the pattern I suspect may become more obvious.

What if $T_3(c) = T_2(c)+2\cdot\binom{c+2}{5}$, and $T_4(c) = T_3(c)+2\cdot\binom{c+3}{7}$,

the pattern being $T_d(c) = T_{d-1}(c)+2\cdot\binom{c+d-1}{2d-1}$, or $T_d(c) = 2\cdot\sum_{i=1}^{d}\binom{c+i-1}{2i-1}$..?

I would like to close this paper with a conjecture, to entice further exploration yielding more interesting properties, or even providing a definitive answer…

Conjecture

Table 6. Maximum number of segments in *d*-tori, after *c* cuts (established in orange, conjectured in yellow), for *c* > 0 and *d* > 0

$T_1(c)$	$2\cdot\binom{c+0}{1}$	$2c$
$T_2(c)$	$T_1(c) + 2\cdot\binom{c+1}{3}$	$\frac{c^3+5c}{3}$
$T_3(c)$	$T_2(c) + 2\cdot\binom{c+2}{5}$	$\frac{c^5+15c^3+104c}{60}$
$T_4(c)$	$T_3(c) + 2\cdot\binom{c+3}{7}$	$\frac{c^7+28c^5+679c^3+4332c}{2520}$
…	…	…
$\mathbf{T_d(c) = 2\cdot\sum_{i=1}^{d}\binom{c+i-1}{2i-1}}$		